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Addendum to vector calculus: fields as co-chains of differential operators

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Abstract. Let $\mathcal{D}$ be the set of differential operators on functions defined on $\mathbb{R}^3$. Let $\alpha^{(p)}$, $p = 0, 1, 2$ and 3 be scalar, vector, pseudovector and density fields on $\mathbb{R}^3$, respectively. Then $\alpha^{(p)}$ can be regarded as a multilinear map $\mathcal{D}^p \to \mathcal{D}$.

Let $r_0, \ldots, r_p \in \mathbb{R}^3$ and let $(r_0, \ldots, r_p)$ be the Euclidean p-simplex having these vertices. Then the integral of $\alpha^{(p)}$ over $(r_0, \ldots, r_p)$ is a function $F^{\alpha^{(p)}}(r_0, \ldots, r_p)$ which satisfies $F^{d\alpha} = \delta_{\text{AS}} F^{\alpha}$. Here $d\alpha^{(p)}$ means grad $\alpha^{(0)}$, curl $\alpha^{(1)}$, or div $\alpha^{(2)}$, and δ_{AS} is a natural cohomological operator (the usual Alexander–Spanier co-boundary operator, given by (5) in the text). For any function $F(r_0, \ldots, r_p)$ there is a natural map $\Phi^F : \mathcal{D}^p \to \mathcal{D}$ (given by (11) in the text) which satisfies $\Phi^{\delta_{\text{AS}} F} = \delta_H \Phi^F$ where δ_H is the Hochschild co-boundary (equation (3)) for co-chains Φ^F on $\mathcal{D}$.

Thus, when $\alpha^{(p)}$ is regarded as being the cochain $\Phi^{F^{\alpha^{(p)}}}$ on $\mathcal{D}$, grad, curl and div all become Hochschild co-boundary operators: grad $\alpha^{(0)}(H)$ is the commutator of the operator H with the function $\alpha^{(0)}$ and curl $\alpha^{(1)}(H_1, H_2)$ measures the amount by which $\alpha^{(1)}$ fails to be a derivation on $\mathcal{D}$. If div $\alpha^{(2)} = 0$ then $\alpha^{(2)}$ provides a deformation of the composition product on $\mathcal{D}$.

This new viewpoint of fields as operator-valued maps of p-tuples of operators has implications in several areas of physics and mathematics. One consequence is that the Hamiltonian in quantum mechanics may be regarded as its own probability current density operator. Another is that Maxwell's equations describe the algebraic character of the electric and magnetic fields E and B regarded as co-chains on $\mathcal{D}$.

We give some explicit formulae for $\alpha^{(p)}(H_1, \ldots, H_p)$.

1. Introduction

There is an isomorphism between the de Rham cohomology of forms on any manifold M and the $\mathcal{F}$-relative Hochschild cohomology of p-co-chains on the associative algebra of differential operators on $C^\infty(M, \mathbb{R})$. For $M = \mathbb{R}^3$ the formulae simplify greatly and are accessible from vector calculus without any differential geometry. There are interesting algebraic interpretations of several physical quantities.

Let $\alpha^{(p)}$, $p = 0, 1, 2$ and 3 be scalar, vector, pseudovector and density fields, respectively, on $\mathbb{R}^3$. (We discuss the case $\mathbb{R}^3 \backslash A$, where $A \subset \mathbb{R}^3$ is a closed subset, later.) Let $\mathcal{D} = \mathcal{D}(\mathbb{R}^3)$ be the associative algebra of differential operators on $C^\infty(\mathbb{R}^3, \mathbb{R}) \equiv \mathcal{F}$, and let $H_j \in \mathcal{D}$, $j = 1, 2, \ldots$.

$d\alpha^{(p)}$ denote grad $\alpha^{(0)}$, curl $\alpha^{(1)}$ and div $\alpha^{(2)}$.

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We may regard $\alpha^{(p)}$ as a differential p-form, that is an antisymmetric $\mathbb{R}$-linear function of p vector fields. Given vector fields X_j, (which may be regarded as first-order differential operators, and so elements of $\mathcal{D}$), we can make

$$\alpha^{(p)}(X_1, \ldots, X_p)$$

which is a function and so may be regarded as a zero-order differential

The present work extends the domain of $\alpha^{(p)}$ from p-tuples of first-order operators X_j to p-tuples of differential operators $H_j \in \mathcal{D}$ of any order. The extended $\alpha^{(p)}$ is denoted $D^{\alpha^{(p)}}$; its action on the H_j yields a differential operator rather than a function:

$$D^{\alpha^{(p)}}(H_1, \ldots, H_p) \in \mathcal{D}. \tag{1}$$

Such maps from $\otimes^p\mathcal{D}$ to $\mathcal{D}$ are called Hochschild p-co-chains on $\mathcal{D}$. The 0-co-chains are the elements of $\mathcal{D}$. The important property of D^α is that

$$D^{d\alpha} = \delta_{\mathrm{H}} D^\alpha \tag{2}$$

where δ_{H}, the Hochschild co-boundary on co-chains on $\mathcal{D}$, is given by

$$\begin{aligned}\delta_{\mathrm{H}} D^{\alpha^{(p)}}(H_1, \ldots, H_{p+1}) &= H_1 D^{\alpha^{(p)}}(H_2, \ldots, H_{p+1}) - D^{\alpha^{(p)}}(H_1 H_2, \ldots, H_{p+1}) + \cdots \\ &\quad + (-1)^p D^{\alpha^{(p)}}(H_1, \ldots, H_p H_{p+1}) + (-1)^{p+1} D^{\alpha^{(p)}}(H_1, \cdots, H_p) H_{p+1}. \end{aligned} \tag{3}$$

Thus, for example, the scalar field $\alpha^{(0)}$ is a zero-order differential operator

$$D^{\alpha^{(0)}} = \alpha^{(0)} \in \mathcal{D}.$$

The vector field $\alpha^{(1)}$ can be regarded as a map $D^{\alpha^{(1)}} : \mathcal{D} \to \mathcal{D}$, and (2) states that

$$\begin{aligned}&D^{\operatorname{grad} \alpha^{(0)}}(H) = \left[H, \alpha^{(0)}\right] \\ &D^{\operatorname{curl} \alpha^{(1)}}(H_1, H_2) = H_1 \circ D^{\alpha^{(1)}}(H_2) - D^{\alpha^{(1)}}(H_1 H_2) + D^{\alpha^{(1)}}(H_1) \circ H_2 .\end{aligned}$$

Thus a curl-free vector field $\alpha^{(1)}$ gives a derivation $D^{\alpha^{(1)}}$: $\mathcal{D} \to \mathcal{D}$ which is inner if $\alpha^{(1)} = \operatorname{grad} \alpha^{(0)}$. Similarly a divergence-free pseudovector field $\alpha^{(2)}$ gives a deformation of the algebra $\mathcal{D}$, which is trivial if $\alpha^{(2)}$ is a curl. This aspect is pursued in section 6 where we shall interpret Maxwell's equations as algebraic statements about the action of the Hochschild 1-co-chains E and $*B$ and the 2-co-chains $*E$ and B on the algebra $\mathcal{D}$.

The theory can be extended to map p-forms on any manifold M into Hochschild p-co-chains on $\mathcal{D}(M)$ [1] and in fact leads to an isomorphism between the de Rham and $\mathcal{F}$-relative Hochschild cohomologies.

When restricted back to vector fields, D^α does not quite agree with α:

$$D^\alpha(X_1, \ldots, X_p) = \frac{1}{p!}\alpha(X_1, \ldots, X_p).$$

The factor $1/p!$ could be eliminated by rescaling the conventional definition (3) of δ_{H}.

The construction of D^α from α presented in section 2 is more transparent than that described in [1]. We first obtain the integral $F^{\alpha^{(p)}}(r_0, \ldots, r_p)$ of $\alpha^{(p)}$ over the Euclidean p-simplex having vertices $r_0, \ldots, r_p$. We then make from $F^{\alpha^{(p)}}$ the required multilinear map $D^{\alpha^{(p)}}$: $\mathcal{D}^p \to \mathcal{D}$.

Section 3 describes the main properties of D^α. Section 4 contains some formulae for the maps $D^{\alpha^{(p)}}$. In section 5 we offer a geometrical interpretation of the map $D^{\alpha^{(1)}}$: $\mathcal{D} \to \mathcal{D}$ and relate it in secuion 6 to the probability current in quantum mechanics. Section 6 also contains the algebraic interpretation of Maxwell's equations.

2. Fields as co-chains on $\mathcal{D}$

The construction of $D^{\alpha^{(p)}}$ proceeds in two stages. In stage one we make the function $\alpha^{(p)}$ from

$$F^{\alpha^{(p)}}(r_0,\ldots,r_p)=\int_{(r_0,\ldots,r_p)}\alpha^{(p)}(r)\,\mathrm{d}^p r \tag{4}$$

where $(r_0,\ldots,r_p)$ is the Euclidean p-simplex having $r_0,\ldots,r_p$ as vertices. For $p=1$, (r_0,r_1) is the directed straight line segment from r_0 to r_1. Similarly, (r_0,r_1,r_2) is an oriented triangle, and $(r_0,\ldots,r_3)$ is an oriented tetrahedron. The p-simplex $(r_0,\ldots,r_p)$ has boundary

$$\partial(r_0,\ldots,r_p)=\sum_{j=0}^{p}(-1)^j(r_0,\ldots,\hat{r}_j,\ldots,r_p)$$

where $\hat{}$ denotes omission.

Let us define the (Alexander–Spanier co-boundary) operator δ_{AS} by

$$\delta_{\mathrm{AS}}(F)=F\circ\partial$$

i.e.

$$\delta_{\mathrm{AS}}\big(F^{\alpha^{(p)}}\big)(r_0,\ldots,r_{p+1})=\sum_{j=0}^{p+1}(-1)^j F^{\alpha^{(p)}}(r_0,\ldots,\hat{r}_j,\ldots,r_{p+1})\,. \tag{5}$$

Then by Stokes's theorem

$$F^{\mathrm{d}\alpha^{(p)}}=\delta_{\mathrm{AS}}F^{\alpha^{(p)}}\,. \tag{6}$$

So in stage one we have converted a field $\alpha^{(p)}$ into a function $F^{\alpha^{(p)}}$ of $p+1$ variables $r_0,\ldots,r_p$.

In stage two we convert a function $F(r_0,\ldots,r_p)$ into a Hochschild p-co-chain Φ^F on $\mathcal{D}$ as follows. For a *separable* function F,

$$F(r_0,\ldots,r_p)=f_0(r_0)f_1(r_1)\ldots f_p(r_p)\qquad\text{i.e.}\quad F=f_0\otimes\ldots\otimes f_p \tag{7}$$

and $H_j\in\mathcal{D}$, define

$$\Phi^F(H_1,\ldots,H_p)=f_0H_1f_1H_2\ldots H_pf_p\ \in\mathcal{D}\,. \tag{8}$$

Note that

$$\begin{aligned}\delta_{\mathrm{AS}}(f_0\otimes\ldots\otimes f_p)&=1\otimes f_0\otimes\ldots\otimes f_p-f_0\otimes1\otimes f_1\otimes\ldots\otimes f_p+\cdots\\&\quad+(-1)^{p+1}f_0\otimes\ldots\otimes f_p\otimes1\end{aligned} \tag{9}$$

and that for F given by (7)

$$\Phi^{\delta_{\mathrm{AS}}F}=\delta_{\mathrm{H}}\Phi^F\,. \tag{10}$$

Every function $F(r_0,\ldots,r_p)$ is the limit of a sum of separable functions of the form (7). We therefore define for a general F and any function $\psi\in\mathcal{F}$

$$\begin{aligned}(\Phi^F(H_1,\ldots,H_p)\psi)(r_0)\\=[H_1(r_1)[H_2(r_2)[\ldots H_p(r_p)[F(r_0,\ldots,r_p)\psi(r_p)]]_{r_p=r_{p-1}}]\ldots]_{r_1=r_0}\end{aligned} \tag{11}$$

since this formula reduces to (8) in the case when F is separable, and so satisfies (10). Here the notation $H_1(r_1)$ means, for example, that if

$$H_1(r)=a^{ij}(r)\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$$

then

$$H_1(r_1) = a^{ij}(r_1)\frac{\partial}{\partial x_1^i}\frac{\partial}{\partial x_1^j}\,.$$

Equation (11) depends only on the partial derivatives of F at the diagonal point $r_p = r_{p-1} = \cdots = r_0$ of $(\mathbb{R}^3)^{p+1}$. We define

$$D^{\alpha^{(p)}} = \Phi^{F^{\alpha^{(p)}}}\,. \tag{12}$$

The main result, equation (2), follows from (6) and (10).

The definition (12) will make sense even when $\mathbb{R}^3$ is replaced by any open subset $\mathbb{R}^3\backslash A$ since although $F^{\alpha^{(p)}}$ will not now be *globally* definable, we can still define it by (4) for $r_1, \ldots, r_p$ in a small open contractible neighbourhood of r_0. This suffices to define the partial derivatives required in (11)†.

3. Some properties of D^α

(i) It follows from (4) of F^α that

$$F^{\alpha^{(p)}}(r_0, \ldots, r_p) = 0$$

whenever $r_j = r_{j-1}$, $1 \leqslant j \leqslant p$. For such F in (11), the expression

$$[H_p(r_p)[F(r_0, \ldots, r_p)\psi(r_p)]]_{r_p=r_{p-1}} \tag{13}$$

will vanish if the differential operator H_p has order zero. The factor F must be differentiated by H_p at least once before r_p is set equal to r_{p-1}, in order not to vanish. So H_p may differentiate ψ at most ord $H_p - 1$ times. Similar arguments for the other H_j imply that the order of the operator $\Phi^F(H_1, \ldots, H_p)$ is

$$\operatorname{ord}\Phi^F(H_1, \ldots, H_p) = \sum_{j=1}^{p}\operatorname{ord}H_j - p$$

and

$$\Phi^F(H_1, \ldots, H_p) = 0$$

if any H_j has order zero.

(ii) Let g_j, $j = 1, 2, \ldots$ be functions. Then if F is separable, (7),

$$\Phi^F(g_1H_1, g_2H_2, \ldots, H_pg_3) = g_1 \circ \Phi^F(H_1 \circ g_2, H_2, \ldots, H_p) \circ g_3$$

and this property extends by linearity to Φ^F for any F, not necessarily separable.

(iii) Define the conjugate operators

$$g^* = g \qquad \left(\frac{\partial}{\partial x^i}\right)^* = -\frac{\partial}{\partial x^i} \qquad (H_1H_2)^* = H_2^*H_1^*\,.$$

† Formula (9) superficially resembles the equation for δ in the description of the differential envelope $\Omega(\mathcal{F})$ using functions $F(r_0, \ldots, r_p)$ in non-commutative differential geometry, [2]. One difference is that in the present work

$$F^{f_0\mathrm{d}f_1}(r_0, r_1) \neq f_0(r_0)F^{\mathrm{d}f_1}(r_0, r_1)$$

whilst in the differential envelope

$$F^{f_0\delta f_1}(r_0, r_1) = f_0(r_0)(f_1(r_1) - f_1(r_0)) = f_0(r_0)F^{\delta f_1}(r_0, r_1)\,.$$

Then

$$D^{\alpha^{(p)}}(H_1, \ldots, H_p)^* = (-1)^{p(p+1)/2} D^{\alpha^{(p)}}(H_p^*, \ldots, H_1^*) . \tag{14}$$

To see this, put

$$F^*(r_0, \ldots, r_p) = F(r_p, \ldots, r_0) .$$

So for separable F,

$$\Phi^F(H_1, \ldots, H_p)^* = f_p H_p^* \ldots H_1^* f_0 = \Phi^{F^*}(H_p^*, \ldots, H_1^*) .$$

This property extends by linearity to Φ^F for any F. Since $F^{\alpha^{(p)}}(r_0, \ldots, r_p)$ is totally antisymmetric,

$$F^{\alpha^{(p)}*} = (-1)^{p(p+1)/2} F^{\alpha^{(p)}}$$

giving (14).

4. Some explicit calculations

For $p = 1$, the line segment (r_0, r_1) may be parametrized as

$$r(t) = r_0 + t(r_1 - r_0) \qquad 0 \leqslant t \leqslant 1 \qquad \frac{\partial x^i}{\partial t} = x_1^i - x_0^i .$$

Hence

$$F^{\alpha^{(1)}}(r_0, r_1) = \int_{t=0}^{1} \mathrm{d}t\, \alpha_i^{(1)}(r(t))\big(x_1^i - x_0^i\big) .$$

Similarly for $p = 2$,

$$r(t_1, t_2) = r_0 + t_1(r_1 - r_0) + t_2(r_2 - r_1) \qquad 0 \leqslant t_2 \leqslant t_1 \leqslant 1$$

$$F^{\alpha^{(2)}}(r_0, r_1, r_2) = \int_{t_1=0}^{1} \mathrm{d}t_1 \int_{t_2=0}^{t_1} \mathrm{d}t_2\, \alpha_{ij}^{(2)}(r(t_1, t_2))\big(x_1^i - x_0^i\big)\big(x_2^j - x_1^j\big) .$$

Here it is convenient to write the pseudovector components $\alpha_{23}^{(2)} = -\alpha_{32}^{(2)} = \alpha_1^{(2)}$, and so on. Denote

$$F_{,2^a 2^b}(r_0, r_1, r_1) = \left[\frac{\partial}{\partial x_2^a}\frac{\partial}{\partial x_2^b} F(r_0, r_1, r_2)\right]_{r_2=r_1} .$$

Then

$$F_{,1^a}^{\alpha^{(1)}}(r_0, r_0) = \alpha_a^{(1)}(r_0) \qquad F_{,1^a 1^b}^{\alpha^{(1)}}(r_0, r_0) = \tfrac{1}{2}\big(\alpha_{a,b}^{(1)} + \alpha_{b,a}^{(1)}\big)(r_0)$$

$$F_{,2^b}^{\alpha^{(2)}}(r_0, r_1, r_1) = \big(x_1^i - x_0^i\big) \int_{t_1=0}^{1} \mathrm{d}t_1 \int_{t_2=0}^{t_1} \mathrm{d}t_2\, \alpha_{ib}^{(2)}(r(t_1, t_2))$$

$$\left[\frac{\partial}{\partial x_1^a} F_{,2^b}^{\alpha^{(2)}}(r_0, r_1, r_1)\right]_{r_1=r_0} = \tfrac{1}{2}\alpha_{ab}^{(2)}(r_0)$$

$$\left[\frac{\partial}{\partial x_1^a} F_{,2^b 2^c}^{\alpha^{(2)}}(r_0, r_1, r_1)\right]_{r_1=r_0} = \tfrac{1}{6}\big(\alpha_{ab,c}^{(2)} + \alpha_{ac,b}^{(2)}\big)(r_0) .$$

It then follows from (11) that for $H_1 = \partial/\partial x^a$,

$$\big(D^{\alpha^{(1)}}(\partial_a)\psi\big)(r_0) = \left[\frac{\partial}{\partial x_1^a}\big[F^{\alpha^{(1)}}(r_0, r_1)\psi(r_1)\big]\right]_{r_1=r_0} = F_{,1^a}^{\alpha^{(1)}}(r_0, r_0)\psi(r_0) = \alpha_a^{(1)}(r_0)\psi(r_0) .$$

Similarly

$$\begin{aligned}\left(D^{\alpha^{(1)}}(\partial_a\partial_b)\psi\right)(r_0) &= \left[\frac{\partial}{\partial x_1^a}\frac{\partial}{\partial x_1^b}\left[F^{\alpha^{(1)}}(r_0,r_1)\psi(r_1)\right]\right]_{r_1=r_0} \\ &= F^{\alpha^{(1)}}_{,1^a1^b}(r_0,r_0)\psi(r_0)+F^{\alpha^{(1)}}_{,1^a}(r_0,r_0)\psi_{,b}(r_0)+F^{\alpha^{(1)}}_{,1^b}(r_0,r_0)\psi_{,a}(r_0) \\ &= \left[\tfrac{1}{2}\left(\alpha^{(1)}_{a,b}+\alpha^{(1)}_{b,a}\right)+\alpha^{(1)}_a\partial_b+\alpha^{(1)}_b\partial_a\right]\psi(r_0)\,. \end{aligned} \qquad (15)$$

In particular, with $H=\partial_a\partial_a=\nabla^2$,

$$D^{\alpha^{(1)}}(\nabla^2)=\operatorname{div}\alpha^{(1)}+2\alpha^{(1)}\operatorname{grad}.$$

If $\alpha^{(1)}=\operatorname{grad}\alpha^{(0)}$, the right-hand side reduces to the commutator $[\nabla^2,\alpha^{(0)}]$, consistently with (2). One may similarly compute

$$\begin{aligned} D^{\alpha^{(2)}}(\partial_a,\partial_b) &= \tfrac{1}{2}\alpha^{(2)}_{ab} \\ D^{\alpha^{(2)}}(\partial_a,\partial_b\partial_c) &= \tfrac{1}{6}\left(\alpha^{(2)}_{ab,c}+\alpha^{(2)}_{ac,b}\right)+\tfrac{1}{2}\left(\alpha^{(2)}_{ab}\partial_c+\alpha^{(2)}_{ac}\partial_b\right) \\ D^{\alpha^{(2)}}(\nabla^2,\nabla^2) &= \tfrac{2}{3}\alpha^{(2)}_{ab,b}\partial_a=\tfrac{2}{3}\operatorname{curl}\alpha^{(2)}\operatorname{grad}. \end{aligned}$$

The general formulae are as follows. Let I and J, be multi-indices, explicitly $I=(i_1,\ldots,i_{|I|})$, and denote

$$\partial_I=\frac{\partial}{\partial x^{i_1}}\cdots\frac{\partial}{\partial x^{i_{|I|}}}.$$

Then

$$D^{\alpha^{(1)}}(\partial_I)=\mathcal{S}_I\sum_{d=1}^{|I|}\binom{|I|}{d}\alpha^{(1)}_{i_1,i_2\ldots i_d}\partial_{i_{d+1}}\ldots\partial_{i_{|I|}}\,. \qquad (16)$$

$$D^{\alpha^{(2)}}(\partial_I,\partial_J)=\mathcal{S}_I\mathcal{S}_J\sum_{d=1}^{|I|}\sum_{d'=1}^{|J|}\binom{|I|}{d}\binom{|J|}{d'}\frac{d}{d+d'}\alpha^{(2)}_{i_1j_1,i_2\ldots i_dj_2\ldots j_{d'}}\partial_{i_{d+1}}\ldots\partial_{j_{|J|}} \qquad (17)$$

where $\mathcal{S}_I$ denotes symmetrization over I.

5. Meaning of $D^{\alpha^{(1)}}(H)$

The Euclidean metric on $M\equiv\mathbb{R}^3\backslash A$ provides a flat connection ∇_0 on the tangent bundle TM. Then each $H\in\mathcal{D}$ can be uniquely expressed as

$$H(\nabla_0)=\sum_{k=0}^{\operatorname{ord}H}a^{(k)}\nabla_0^k$$

for some symmetric contravariant tensor fields $a^{(k)}$ on M. In detail

$$H\psi=a^{(0)}\psi+a^{(1)i_1}\psi_{;i_1}+a^{(2)i_1i_2}\psi_{;i_1i_2}+\cdots.$$

Equation (16) tells us that

$$\begin{aligned} D^{\alpha^{(1)}}\left(a^{(k)}\nabla_0^k\right)\psi(r_0) &= a^{(k)}(r_0)\left[\nabla_0^k(r_1)\left[\int_{(r_0,r_1)}\alpha^{(1)}(r)\,\mathrm{d}r\,\psi(r_1)\right]\right]_{r_1=r_0} \\ &= a^{(k)}(r_0)\sum_{d=1}^{k}\binom{k}{d}\left(\nabla_0^{d-1}\alpha^{(1)}\right)(r_0)\left(\nabla_0^{k-d}\psi\right)(r_0) \end{aligned}$$

so that

$$D^{\alpha^{(1)}}\big(a^{(k)}\nabla_0^k\big) = a^{(k)}\big(\alpha^{(1)}\circ\nabla_0^{k-1} + \nabla_0\circ\alpha^{(1)}\circ\nabla_0^{k-2} + \cdots + \nabla_0^{k-1}\circ\alpha^{(1)}\big)$$
$$= a^{(k)}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(\big(\nabla_0+\epsilon\alpha^{(1)}\big)^k - \nabla_0^k\big). \qquad (18)$$

Hence we may write

$$D^{\alpha^{(1)}}(H) = \left(\frac{\partial H(\nabla)}{\partial\nabla}, \alpha^{(1)}\right)\bigg|_{\nabla=\nabla_0}. \qquad (19)$$

That is to say $D^{\alpha^{(1)}}(H)$ is the Fréchet derivative of $H(\nabla)$ in the direction $\alpha^{(1)}$ evaluated at the connection ∇_0. Here $H(\nabla)$ is regarded as an operator-valued function on the space of connections on M.

6. Applications

6.1. The probability current in quantum mechanics

Equation (19) is reminiscent of the formula '$\delta\mathcal{H}/\delta A$' for the current in gauge theory [3]. The link is made explicit in this section.

In traditional quantum mechanics the wave function ψ of a particle moving in $\mathbb{R}^3\backslash A = M$ is a square-integrable complex-valued function on M which satisfies

$$\mathrm{i}\partial_t\psi = H\psi$$

where $H \in \mathcal{D}$ is Hermitian. The probability density $\rho = \bar\psi\psi$ then satisfies

$$\int \partial_t\rho = 2\,\mathrm{Im}\langle\psi, H\psi\rangle = 0$$

which implies that we may write

$$\partial_t\rho = -\mathrm{div}\, \boldsymbol{J}(\psi) \qquad (20)$$

for some probability flux vector $\boldsymbol{J}(\psi)$. Although (20) only fixes the divergence of $\boldsymbol{J}(\psi,)$ the textbooks say that when

$$H = -\tfrac{1}{2}\nabla^2 + V(\boldsymbol{r}) \qquad (21)$$

the correct $\boldsymbol{J}(\psi)$ among all the possible candidates with the right divergence is

$$\boldsymbol{J}(\psi) = \mathrm{Im}(\bar\psi\nabla\psi). \qquad (22)$$

The usual justification is that this gives $\boldsymbol{J} = \boldsymbol{k}$ when $\psi = \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$. This raises the question: what structure is required to select the 'correct' $\boldsymbol{J}$ when H is an arbitrary Hermitian operator?

It is useful to re-pose the problem in terms of de Rham p-currents [4]. A p-current is a real-valued linear function on 'test' p-forms. Thus a 0-current ρ is a scalar density or generalized function, characterized by its action on test functions $f \in \mathcal{F}$,

$$\langle\!\langle \rho, f\rangle\!\rangle = \int_M \rho f \in \mathbb{R}.$$

A 1-current $\boldsymbol{J}$ is a vector density and acts on test 1-forms η (which must be smooth and of sufficiently fast decrease at infinity),

$$\langle\!\langle \boldsymbol{J}, \eta\rangle\!\rangle = \int_M J^i\eta_i.$$

There is a natural map 'div' from p-currents to $(p-1)$-currents:

$$\langle\langle \operatorname{div} \boldsymbol{C},\, \sigma \rangle\rangle = -\langle\langle \boldsymbol{C},\, \mathrm{d}\sigma \rangle\rangle \,.$$

If we multiply (20) by f and integrate, we obtain

$$\begin{aligned}\langle\langle \boldsymbol{J}(\psi),\, \mathrm{d}f \rangle\rangle &= -\langle\langle \operatorname{div} \boldsymbol{J}(\psi),\, f \rangle\rangle = \langle\langle \partial_t \rho,\, f \rangle\rangle = \int \partial_t(\bar{\psi}\psi) f \\ &= \langle \psi,\, \mathrm{i}[f, H]\psi \rangle = -\mathrm{i}\langle \psi,\, D^{\mathrm{d}f}(H)\psi \rangle \,. \end{aligned} \tag{23}$$

Equation (23) tells us that the action of $\mathrm{i}\boldsymbol{J}(\psi)$ on an exact 1-form $\mathrm{d}f$ is the expectation value of the operator $D^{\mathrm{d}f}(H)$. The problem is to extend the definition of $\mathrm{i}\boldsymbol{J}(\psi)$ to any 1-form η. This is the 'current' version of the problem with which we began—to choose $\boldsymbol{J}(\psi)$ given only its divergence. It is natural to conjecture that

$$\langle\langle \mathrm{i}\boldsymbol{J}(\psi),\, \eta \rangle\rangle = \langle \psi,\, D^{\eta}(H)\psi \rangle \,.$$

One may now regard H itself as an operator-valued de Rham current whose action on the test 1-form η gives the operator $D^{\eta}(H)$ whose expectation value in the state ψ is the usual probability current $\mathrm{i}\boldsymbol{J}(\psi)$ smeared by η. In this sense H *is* the probability current. The structure required to select the 'correct' $\boldsymbol{J}$ is that needed to turn forms into co-chains on $\mathcal{D}$, namely the Euclidean metric in the present instance. In the most general case a connection on TM provides the required structure, although other constructions exist [1].

One can check that when H is given by (21),

$$D^{\eta}(H) = -\tfrac{1}{2}(\eta_i \circ \partial_i + \partial_i \circ \eta_i)$$

and

$$\int \bar{\psi} D^{\eta}(\mathrm{i}H)\psi \,\mathrm{d}^3\boldsymbol{r} = -\tfrac{1}{2}\mathrm{i} \int \eta_i(\bar{\psi}\partial_i\psi - (\partial_i\bar{\psi})\psi)\,\mathrm{d}^3\boldsymbol{r} \,.$$

This example can be generalized to create conserved Noether currents for any Hermitian linear differential equation [5].

It may also be adapted to geometric quantization theory. There, H acts on sections of a complex line bundle E over M. The 1-form η acts as a translation $(\Gamma_i \mapsto \Gamma_i + \epsilon\eta_i)$ on the Affine space of connections on E rather than as the translation $\Gamma^k_{ij} \mapsto \Gamma^k_{ij} + \epsilon\eta_i\delta^k_j$ on the connections on TM. See [6].

6.2. *Maxwell's equations*

Maxwell's equations in a vacuum [7]

$$\mathrm{d}B = 0 \tag{24}$$

$$\partial_t B = -\mathrm{d}E \tag{25}$$

$$\mathrm{d}(*E) = 0 \tag{26}$$

$$\partial_t(*E) = \mathrm{d}(*B) \tag{27}$$

may be interpreted as statements about the action of the Hochschild 1-co-chains D^E, D^{*B} and 2-co-chains D^{*E}, D^B on the algebra $\mathcal{D}$. In section 4 we suppressed the symbols $*$ but here we shall make them explicit in order to indicate the degree of the form involved: $(*E)_{23} = -(*E)_{32} = E_1$; $(*B)_1 = B_{23}$. Equations (24) and (26) tell us that D^B and D^{*E} are deformations of the algebra $\mathcal{D}$. That is to say, for B we may define a new composition product $\circ_{\epsilon B}$ on $\mathcal{D}$

$$H_1 \circ_{\epsilon B} H_2 := H_1 \circ H_2 + \epsilon D^B(H_1, H_2) \,. \tag{28}$$

One can check that

$$H_1 \circ_{\epsilon B} (H_2 \circ_{\epsilon B} H_3) - (H_1 \circ_{\epsilon B} H_2) \circ_{\epsilon B} H_3 = \epsilon \delta D^B(H_1, H_2, H_3) + O(\epsilon^2) = O(\epsilon^2)$$

since $\delta D^B = D^{dB} = 0$, so that the new composition law is associative to order ϵ. In particular,

$$\partial_a \circ_{\epsilon B} \partial_b = \partial_a \partial_b + \tfrac{1}{2}\epsilon B_{ab}$$

and so

$$[\partial_a, \partial_b]_{\epsilon B} := \partial_a \circ_{\epsilon B} \partial_b - \partial_b \circ_{\epsilon B} \partial_a = \epsilon B_{ab} .$$

From (18) for any A,

$$H(\nabla_0 + \epsilon A) = H(\nabla_0) + \epsilon D^A(H) + O(\epsilon^2) . \tag{29}$$

So if B has a vector potential A, $B = \mathrm{d}A$, the deformed product (28) becomes

$$\begin{aligned} H_1 \circ_{\epsilon B} H_2 &= H_1 H_2 + \epsilon\big(H_1 D^A(H_2) - D^A(H_1 H_2) + D^A(H_1) H_2\big) \\ &= \big(H_1 + \epsilon D^A(H_1)\big)\big(H_2 + \epsilon D^A(H_2)\big) - \epsilon D^A(H_1 H_2) + O(\epsilon^2) \end{aligned}$$

which from (29) is equal to

$$H_1 H_2 + H_1(\nabla_0 + \epsilon A) H_2(\nabla_0 + \epsilon A) - (H_1 H_2)(\nabla_0 + \epsilon A) + O(\epsilon^2) . \tag{30}$$

Thus the deformed product $H_1 \circ_{\epsilon B} H_2$ arises in this case by transforming the operators $H_j(\nabla_0) \rightarrow H_j(\nabla_0 + \epsilon A)$ in a way determined by the perturbation of the Euclidean connection by the vector potential A. Such deformations may be considered trivial. Any 1-form A will provide a trivially deformed product on $\mathcal{D}$ in this way.

Even when B has no vector potential, equation (25) shows that the time derivative of the deformed product of two time-independent operators H_1, H_2,

$$\partial_t (H_1 \circ_{\epsilon B} H_2) = H_1 \circ_{\epsilon B_t} H_2 \tag{31}$$

will be given by the perturbation $\nabla_0 \rightarrow \nabla_0 - \epsilon E$, and so is trivial in the above sense. This is the algebraic interpretation of (25); the geometric version is of course that B remains in the same de Rham cohomology class as time passes.

In the static case the deformed product $\circ_{\epsilon B}$ is constant in time; $\mathrm{d}E = 0$ so that D^E is a derivation on $\mathcal{D}$.

Equations (26) and (27) yield a similar co-chain interpretation.

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